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Lepton Mixing Parameters from Discrete and CP Symmetries

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Abstract

We consider a scenario with three Majorana neutrinos in which a discrete, finite flavour group G_f is combined with a generalized CP transformation. We derive conditions for consistently defining such a setup and analyze the mathematical structure of the arising group G_{CP}. We show for general G_{CP}, broken to $G_e \subset G_f$ in the charged lepton and to $G_\nu = Z_2 \times CP$ in the neutrino sector, that lepton mixing angles and CP phases (Dirac as well as Majorana) only depend on one single parameter θ which can take values between 0 and 2π. We perform a comprehensive study for $G_f = S_4$ and find five cases which are phenomenologically interesting. They naturally lead to a non-zero reactor mixing angle and all mixing parameters are strongly correlated. Some of the patterns predict maximal atmospheric mixing and maximal Dirac phase, while others predict trivial Dirac and Majorana phases.

1 Introduction

Flavour groups G_f and their peculiar breaking to residual symmetries G_e and G_ν in the charged lepton and in the neutrino sector, respectively, have been applied in the past in order to predict lepton mixing angles and the Dirac phase in a model-independent way, for reviews see [1]. One of the most prominent examples is tri-bimaximal (TB) mixing [2] which can be derived with the help of the flavour groups A_4 [3] and S_4 [4]. Recent measurements of the reactor mixing angle θ_{13}, $\sin\theta_{13} \approx 0.15$, [5, 6], however, clearly show that many patterns which have been discussed in the literature are strongly disfavored, because they predict vanishing or very small θ_{13}. For this reason, proposals have been made in which either the groups G_f are chosen to be large, e.g. $\Delta(96)$ and $\Delta(384)$ [7], or the symmetry is reduced, e.g. $G_\nu = Z_2$ is considered instead of $G_\nu = Z_2 \times Z_2$ in the neutrino sector [8].

We explore a different approach in this paper and consider a scenario with three Majorana neutrinos in which a discrete, finite flavour group G_f and a CP symmetry are combined and are broken in such a way that the residual symmetry in the neutrino sector is $G_\nu = Z_2 \times CP$ with Z_2 being a subgroup of G_f. The residual symmetry $G_e \subset G_f$ in the charged lepton sector is - as in preceding approaches [4, 7, 8] - chosen as cyclic symmetry (or product thereof) which allows to distinguish between the three generations of charged leptons. We show for a general G_f that such a breaking pattern allows to predict lepton mixing angles and Dirac as well as Majorana phases in terms of a single real parameter θ. A non-vanishing reactor mixing angle can be easily accommodated and at the same time relations between mixing angles and CP phases are obtained. In contrast to the scenario without a CP symmetry, we also predict Majorana phases with our present approach.

We discuss in detail the conditions which have to be fulfilled in order to consistently formulate a setup with a flavour symmetry G_f and a CP symmetry and in order to define the group G_ν as direct product of Z_2 and CP. We also show that in general several independent CP transformations might be compatible with a flavour group G_f which lead, in general, to physically different results. The mathematical structure of the group arising from the combination of G_f and CP turns out to be a semi-direct product of the form $G_{CP} = G_f \rtimes H_{CP}$ with H_{CP} being the group associated with CP. We comment on particular cases in which this product becomes a direct one. We exemplify our formalism with a comprehensive study of $G_f = S_4$ and show several interesting mixing patterns whose predictions for the mixing angles are close to the best fit results [6] for certain values of the parameter θ.

The possibility to combine a flavour symmetry G_f with a CP symmetry is not new and has been already discussed in some cases in the literature [9–12]. An interesting example is the so-called $\mu\tau$ reflection symmetry which is a combination of the canonical CP transformation and the $\mu\tau$ exchange symmetry. This generalized CP transformation permutes a muon neutrino (antineutrino) and a tau antineutrino (neutrino). If imposed on the neutrino mass matrix (in the basis in which the charged lepton mass matrix is diagonal), it forces the elements of the second and third rows of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix to have the same absolute values. As a consequence, one finds $\sin\theta_{23} = \cos\theta_{23}$ and $\sin\theta_{13}\sin 2\theta_{12}\cos\delta = 0$. Thus, maximal atmospheric mixing is predicted and, in view of the latest global fits [6], also the Dirac phase δ has to be maximal.

Recently, also the combination of the flavour group S_4 and a certain CP transformation has been discussed in two models [12, 13].

Other contexts in which flavour symmetries and CP violation appear together are: the idea of so-called geometrical CP violation in which the potential of certain scalars is constrained by a flavour group in such a way that their vacuum expectation values spontaneously break CP symmetry with phases independent of the parameters of the Lagrangian [14]; the accidental presence of CP symmetries has been noticed in potentials invariant under (single- and double-valued) dihedral groups [15], while attempts to relate the prediction of CP violation to particular properties of certain representations of the group T' can be found in [16].

The paper is organized as follows: in section 2 we first recall several well-known facts about CP transformations and then discuss their combination with a flavour group G_f and which conditions have to be fulfilled in order to consistently define such a setup. We also comment on the mathematical structure of the group arising from this combination. We then present the results for mixing angles and CP phases for a general G_{CP}, broken to G_e and $G_\nu = Z_2 \times CP$, assuming three generations of Majorana neutrinos. We briefly mention the case of Dirac neutrinos. We furthermore study the possibility and conditions for the presence of an accidental CP symmetry. In section 3 we then present the case $G_f = S_4$ and the different possible CP transformations compatible with all requirements. We show that there are only a few independent - and phenomenologically interesting - cases for which we discuss the results for CP phases and mixing angles in detail. We also find cases in which the neutrino mass matrix is invariant under $\mu\tau$ reflection symmetry (in the charged lepton mass basis). Furthermore, we analyze particular values of the parameter θ for which the symmetry in the neutrino sector is enhanced, $G_\nu = Z_2 \times Z_2 \times CP$, with $Z_2 \times Z_2$ being a subgroup of a finite flavour group containing S_4. We briefly comment on the group structure of S_4 and CP in the various cases and show several results which can also be obtained for $G_f = A_4$. We summarize and conclude in section 4.

2 Framework

In this section we recall some basic properties of CP transformations and of CP invariance in the lepton sector. We then discuss how to combine CP and internal symmetries and we explain how to use CP and flavour symmetries to constrain lepton mixing parameters. Finally, we comment on the possibility of accidental CP symmetries.

2.1 Generalized CP transformations

We discuss the conditions under which the lepton sector is invariant under CP and the consequences of CP invariance for the leptonic phases. In doing so, we focus on the case of Majorana neutrinos and briefly comment on Dirac neutrinos (and the quark sector). The presented results are well-known and can, for example, be also found in [17]. We only repeat them for completeness and in order to settle our notation and conventions.

We can define a CP transformation on a set of fields, collectively denoted by φ, as

$$\varphi'(x) = X\varphi^*(x_{CP}) \tag{1}$$

in matrix notation and with $x_{CP} = (x^0, -\vec{x})$. The transformation X can be chosen as unitary symmetric matrix,

$$XX^\dagger = XX^* = \mathbb{1} \quad , \tag{2}$$

so that $CP^2 = 1$. We refer to the transformation in eq. (1) as a generalized CP transformation [18]. For spinors we use a two-component notation and omit the obvious action of CP on the spinor indices (which contains suitable phases such that $CP^2 = 1$). In the following we also omit the dependence of the fields on the space-time point x.

Let us consider a generalized CP transformation acting on the three generations of lepton doublets l

$$l' = Xl^* \tag{3}$$

which fulfills eq.(2). In the interaction basis gauge interactions are CP conserving, while the requirement of CP invariance of the Yukawa interactions, including terms giving rise to neutrino masses, constrains both the charged lepton mass matrix m_l[1] and the neutrino mass matrix m_ν:[2]

$$X^* m_l^\dagger m_l X = (m_l^\dagger m_l)^* \quad , \qquad\qquad X m_\nu X = m_\nu^* \quad . \tag{4}$$

In order to study the consequences of CP invariance on the lepton mixing parameters, we diagonalize the mass matrices by unitary matrices U_e and U_ν

$$m_l^\dagger m_l = U_e (m_l^\dagger m_l)^{diag} U_e^\dagger \quad , \qquad\qquad m_\nu = U_\nu^* m_\nu^{diag} U_\nu^\dagger \quad , \tag{5}$$

so that the invariance conditions in eq. (4) become

$$X_e^* (m_l^\dagger m_l)^{diag} X_e = (m_l^\dagger m_l)^{diag} \quad , \qquad\qquad X_\nu m_\nu^{diag} X_\nu = m_\nu^{diag} \quad , \tag{6}$$

with

$$X_e = U_e^\dagger X U_e^* \quad , \qquad\qquad X_\nu = U_\nu^\dagger X U_\nu^* \quad . \tag{7}$$

Notice X_e and X_ν represent the CP transformation X in the mass basis. For non-degenerate charged lepton and neutrino masses one can show that the general solutions of eq. (6) for X_e and X_ν are diagonal matrices

$$X_e = \begin{pmatrix} e^{ix_1} & 0 & 0 \\ 0 & e^{ix_2} & 0 \\ 0 & 0 & e^{ix_3} \end{pmatrix} \quad , \qquad\qquad X_\nu = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix} \tag{8}$$

with

$$x_i \in [0, 2\pi) \quad , \qquad\qquad s_i = \pm 1 \quad . \tag{9}$$

This can be stated differently: the lepton sector is invariant under a generalized CP transformation described by X, if one can find matrices X_e and X_ν of the form as in eqs.(8, 9), such that

$$U_\nu X_\nu U_\nu^T = U_e X_e U_e^T \quad . \tag{10}$$

[1]We use a basis in which right-handed (left-handed) fields are on the left-hand (right-hand) side of the charged lepton mass matrix m_l.

[2]Note that our constraint on the neutrino mass matrix differs by a sign from that in [17]. This is due to a different definition of the action of CP on the spinor indices.

Provided that such X_e and X_ν exist, we can study the consequences for the PMNS matrix

$$U_{PMNS} = U_e^\dagger U_\nu \ . \tag{11}$$

The standard parametrization of the latter is

$$U_{PMNS} = \tilde{U} \ \mathrm{diag}(1, e^{i\alpha/2}, e^{i(\beta/2+\delta)}) \quad , \tag{12}$$

with $\tilde{U}$ being of the form of the Cabibbo-Kobayashi-Maskawa (CKM) matrix V_{CKM} [19]

$$\tilde{U} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad . \tag{13}$$

We use the shorthand notation $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$. The mixing angles θ_{ij} range from 0 to $\pi/2$, while the Majorana phases α, β as well as the Dirac phase δ take values between 0 and 2π.

From eqs.(10,11) we see that

$$X_e^* U_{PMNS} X_\nu = U_{PMNS}^\star \tag{14}$$

holds. If eqs.(8,9) are fulfilled, eq.(14) is simplified to [20]

$$U_{PMNS,ij} e^{-ix_i} s_j = U^*_{PMNS,ij} \tag{15}$$

and the Jarlskog invariant J_{CP} [21]

$$\begin{aligned} J_{CP} &= \mathrm{Im}\left[U_{PMNS,11}U^*_{PMNS,13}U^*_{PMNS,31}U_{PMNS,33}\right] \\ &= \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta \end{aligned} \tag{16}$$

and $\sin\delta$ vanish,[3]

$$J_{CP} = 0 \quad , \quad \sin\delta = 0 \quad . \tag{17}$$

Similar to the Jarlskog invariant, invariants for the two Majorana phases α and β can be defined [22] (see also [23–25]; see [11] for invariants in terms of neutrino mass matrix elements)

$$I_1 = \mathrm{Im}[U^2_{PMNS,12}(U^*_{PMNS,11})^2] = s_{12}^2 c_{12}^2 c_{13}^4 \sin\alpha \quad , \tag{18}$$

$$I_2 = \mathrm{Im}[U^2_{PMNS,13}(U^*_{PMNS,11})^2] = s_{13}^2 c_{12}^2 c_{13}^2 \sin\beta \quad . \tag{19}$$

Also I_1 and I_2 vanish, if the PMNS matrix fulfills eq.(15). Thus, also the Majorana phases α and β are trivial[4]

$$\sin\alpha = 0 \quad , \quad \sin\beta = 0 \quad , \tag{20}$$

[3]The Dirac phase has a physical meaning only if all angles are different from 0 and $\pi/2$, as indicated by the data.

[4]We discard the possibility that $\sin 2\theta_{12} = 0$, $\cos\theta_{13} = 0$ or $\sin 2\theta_{13} = 0$, $\cos\theta_{12} = 0$ are realized. Furthermore, notice that one of the Majorana phases becomes unphysical, if the lightest neutrino mass vanishes.

in this case.

For Dirac neutrinos, m_ν and X_ν are subject to the same constraints as the matrices m_l and X_e, i.e. the neutrino mass matrix and X_ν have to satisfy (only)

$$X_\nu^* (m_\nu^\dagger m_\nu)^{diag} X_\nu = (m_\nu^\dagger m_\nu)^{diag} \tag{21}$$

instead of eq.(4) so that the most general form of X_ν compatible with this equation is one of the same form as X_e in eqs.(8,9) (with x_i called x_i^ν). If this is realized, eq.(14) implies

$$U_{PMNS,ij} e^{-i(x_i - x_j^\nu)} = U^*_{PMNS,ij} \tag{22}$$

and thus a vanishing Jarlskog invariant (and trivial Dirac phase)

$$J_{CP} = 0 \quad , \quad \sin\delta = 0 \quad . \tag{23}$$

These results can also be applied to the quark sector, i.e. the up quark and down quark mass matrices instead of Dirac neutrino and charged lepton mass matrices.

2.2 Generalized CP transformations and flavour symmetries

We now consider a theory that is invariant under both a flavour symmetry G_f and CP. We assume G_f to be a discrete and finite group. However, most of the following statements can also be applied to a continuous symmetry G_f. Scalar and spinor fields transform according to some representation of the flavour group G_f and we denote a set of fields transforming in a generic irreducible representation $\mathbf{r}$ of G_f by φ:

$$\varphi' = A\, \varphi \quad , \tag{24}$$

with A being a unitary matrix depending on the representation $\mathbf{r}$ and on the chosen group element of G_f. Under CP the multiplet φ transforms as

$$\varphi' = X \varphi^* \quad , \tag{25}$$

where X is a unitary symmetric matrix that acts on the representation $\mathbf{r}$.[5]

If we perform a CP transformation, followed by a transformation of G_f and another CP transformation we end up with $\varphi' = (X^{-1} A X)^* \varphi$. By consistency this should be a transformation of G_f for the representation $\mathbf{r}$, for some group element A':

$$(X^{-1} A X)^* = A' \quad . \tag{26}$$

Given the group G_f, this equation constrains the form of X and we would like to determine its general solution that also satisfies eq. (2).

A few remarks are in order: at least one X, namely the canonical CP transformation, exists, if the representation $\mathbf{r}$ is real. It is sufficient to check whether a transformation X

[5]It might happen that it is not possible to define the action of CP on a single irreducible representation of G_f. In this case φ of eq. (25) denotes the smallest combination of irreducible representations of G_f on which the action of CP is well defined.

satisfies the constraint in eq. (26) for a set of generators A_i of the group G_f, because if it does so, then it does it automatically for all elements of the group.[6] The element A' has to have the same order as the element A. If G_f is abelian, all irreducible representations are one-dimensional and the constraint in eq. (26) is satisfied by $X = e^{i\theta}$, with any phase θ. Indeed, A^* in this case is always A^{-1}. If G_f is non-abelian, there are special bases in which we can recognize in a simple way whether there exists (at least) one transformation X which fulfills the constraint in eq. (26), for instance a basis in which all non-diagonal generators are real.

If we perform a change of basis with a unitary matrix Ω in the field space

$$\tilde{\varphi} = \Omega^\dagger \varphi \quad , \tag{27}$$

the unitary matrices X and A transform as

$$\tilde{X} = \Omega^\dagger X \Omega^* \quad , \qquad \tilde{A} = \Omega^\dagger A \Omega \quad , \tag{28}$$

as can be seen using eqs. (24, 25). The constraints in eqs. (2, 26) are covariant under such a transformation Ω. A change of basis can be useful in order to reach a basis in which the action of some elements of G_f and/or of CP is particularly simple. For example, we can use the result that any unitary symmetric matrix X can be written as the product $X = \Omega\Omega^T$ of a unitary matrix Ω with its transpose, in order to change to a basis in which $\tilde{X} = \mathbb{1}$, see eq. (28). At the same time, the constraint in eq. (26) reads $\tilde{A}^* = \tilde{A}'$. In such a basis the action of CP is canonical.

Given a solution X of the constraint in eq. (26) and assuming that $\mathbf{r}$ is a faithful representation[7] of G_f (which does not need to be discrete and/or finite in the following), we would like to know which group arises by combining the transformations of G_f and those of H_{CP}, the parity group generated by CP. In order to do so we enlarge the field space from φ to $\Phi = (\varphi, \varphi^*)^T$. In this space the actions of G_f and CP are given by:

$$\Phi' = \mathcal{A}\Phi \qquad \qquad \Phi' = \mathcal{X}\Phi \quad , \tag{29}$$

with

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \qquad \qquad \mathcal{X} = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \quad , \tag{30}$$

respectively. The unitary matrices $\mathcal{A}$ generate a group which is isomorphic to G_f. The matrix $\mathcal{X}$ satisfies

$$\mathcal{X}^2 = \mathbb{1} \quad , \tag{31}$$

since X is unitary and symmetric, and thus generates a group $\{\mathcal{X}, \mathbb{1}\}$ isomorphic to H_{CP}. We do not distinguish between two isomorphic groups and we use the same notation for both. The consistency condition in eq. (26) reads

$$\mathcal{X}^{-1}\mathcal{A}\mathcal{X} = \mathcal{A}' \quad . \tag{32}$$

[6]The application of a similarity transformation and complex conjugation preserves the usual (matrix) multiplication rules.

[7]In this way, each element of the abstract group G_f is represented by a different representation matrix.

The group G_{CP} we are looking for is that generated by the unitary matrices $\mathcal{A}$ and $\mathcal{X}$. We observe that the closure of this set of transformations is guaranteed by eqs. (31,32). Indeed by using these equations we can show that any element of G_{CP} can be cast into the form $\mathcal{A}\mathcal{H}$ with $\mathcal{A}$ belonging to G_f and $\mathcal{H}$ belonging to the group H_{CP}. Such a decomposition is unique, namely $\mathcal{A}\mathcal{H} = \mathcal{A}'\mathcal{H}'$ implies $\mathcal{A}' = \mathcal{A}$ and $\mathcal{H}' = \mathcal{H}$. This is equivalent to the statement that the intersection of the two groups G_f and H_{CP} is only the neutral element. The multiplication rule of two elements of G_{CP} follows from eqs. (31,32) and is given by

$$\mathcal{A}\mathcal{H}\ \mathcal{A}'\mathcal{H}' = \mathcal{A}\mathcal{A}''\ \mathcal{H}\mathcal{H}' \qquad\qquad \mathcal{A}'' = \mathcal{H}\mathcal{A}'\mathcal{H} \quad . \tag{33}$$

Therefore the group G_{CP} is isomorphic to the semi-direct product of G_f and H_{CP}, $G_{CP} = G_f \rtimes H_{CP}$. G_f is a normal subgroup of G_{CP}, while H_{CP} is in general only a subgroup. If G_f is a finite group, the order of G_{CP} is twice the order of G_f. The semi-direct product reduces to a direct one if and only if $\mathcal{X}$ commutes with all elements of G_f, that is when the condition in eq. (32) is satisfied with $\mathcal{A}' = \mathcal{A}$, for all elements $\mathcal{A}$ of G_f. (This is equivalent to the case in which the condition in eq. (26) is satisfied with $A' = A$, for all A.) Then the subgroup H_{CP} is also normal. We note that under a change of basis

$$\tilde{\Phi} = \mathcal{O}^\dagger \Phi \qquad\qquad \mathcal{O} = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega^* \end{pmatrix} \quad , \tag{34}$$

with $\mathcal{O}$ unitary, all elements of G_{CP} have the same transformation properties, as deduced from eqs.(27,28,30),

$$\tilde{\mathcal{X}} = \mathcal{O}^\dagger \mathcal{X} \mathcal{O} \quad , \quad \tilde{\mathcal{A}} = \mathcal{O}^\dagger \mathcal{A} \mathcal{O} \quad . \tag{35}$$

2.3 Lepton mixing from G_{CP}

One reason for imposing a flavour symmetry G_f is to constrain the form of the lepton mixing matrix in order to explain the observed pattern of mixing angles. A particular approach [26] is to assume the invariance under G_f to be broken in such a way that the combination $m_l^\dagger m_l$ and the neutrino mass matrix m_ν possess residual discrete symmetries G_e and G_ν, respectively. If we consider three generations of Majorana neutrinos, we are naturally led to the choice $G_\nu = Z_2 \times Z_2$, the largest symmetry of m_ν leaving neutrino masses unconstrained [4]. For the group G_e we require the following properties: it should be abelian in order to avoid degeneracies among the charged lepton masses and it should allow to assign different charges to the three different generations, for details see [7]. Thus, G_e is in general a (direct) product of cyclic symmetries, $Z_{m_1} \times \cdots \times Z_{m_p}$. These subgroups of G_f predict the form of the lepton mixing matrix, up to permutations of rows and columns and up to arbitrary Majorana phases [4,7]. The mechanism can be implemented in concrete models in which the desired symmetry breaking pattern of the group G_f can be achieved via spontaneous or explicit breaking and corrections to such a pattern are calculable (and are usually small), see [1] for reviews.

In the present paper we instead consider the case in which the residual symmetry G_ν is $Z_2 \times CP$. As we will see, this allows us to determine all physical phases and mixing angles in terms of a single real parameter θ. A small non-vanishing mixing angle θ_{13} can then be accommodated by a suitable choice of the parameter θ and furthermore testable relations

among the mixing parameters are predicted. Once the flavour group G_f has been chosen, several independent definitions of CP might be possible, leading to physically distinct results. In this subsection we explain the setup and show the general form of the lepton mixing matrix, while we illustrate several interesting features with an explicit example based on the group $G_f = S_4$ in section 3.

We recall that lepton doublets transform in a three-dimensional irreducible representation $\mathbf{r}$ of G_f and that neutrinos are of Majorana type. We further assume that G_f contains a subgroup Z_2 and we denote its generator in the representation $\mathbf{r}$ by Z, $Z^2 = \mathbb{1}$. In order to define the direct product $Z_2 \times CP$, CP should commute with Z_2. Using an argument similar to the one which has led to eq.(26), we can show that the requirement for having a direct product $Z_2 \times CP$ translates into

$$XZ^* - ZX = 0 \quad . \tag{36}$$

Since this equation is covariant under the change of basis given in eqs. (27,28), it is always possible to go to a basis in which Z is diagonal and X canonical. We indicate this particular basis by a hat:

$$Z = \Omega \hat{Z} \Omega^\dagger \quad , \qquad X = \Omega \Omega^T \quad . \tag{37}$$

Barring the trivial case ($\hat{Z} = \pm \mathbb{1}$), we can assume, without loss of generality, that $\hat{Z}$ is of the form

$$\hat{Z} = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{38}$$

In general the conditions of invariance of the neutrino mass matrix m_ν under $Z_2 \times CP$ are

$$Z^T m_\nu Z = m_\nu \quad , \qquad X m_\nu X = m_\nu^* \quad . \tag{39}$$

By making use of eqs. (37) we see that eq.(39) takes the form

$$\hat{Z} (\Omega^T m_\nu \Omega) \hat{Z} = (\Omega^T m_\nu \Omega) \quad , \qquad (\Omega^T m_\nu \Omega) = (\Omega^T m_\nu \Omega)^* \quad . \tag{40}$$

These conditions are satisfied by

$$\Omega^T m_\nu \Omega = \begin{pmatrix} m_{11} & 0 & m_{13} \\ 0 & m_{22} & 0 \\ m_{13} & 0 & m_{33} \end{pmatrix} \quad , \qquad m_{ij} = m_{ij}^* \quad . \tag{41}$$

The original matrix m_ν can be diagonalized by

$$U_\nu^T m_\nu U_\nu = m_\nu^{diag} \quad , \qquad U_\nu = \Omega \; R(\theta) \; K \quad , \tag{42}$$

where $R(\theta)$ is a rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad , \qquad \tan 2\theta = \frac{2 \; m_{13}}{m_{33} - m_{11}} \quad . \tag{43}$$

The unitary matrix K is diagonal with entries ± 1 and $\pm i$ which encode the CP parity of the neutrino states and renders m_ν^{diag} positive (semi-)definite. This fixes the contribution from the neutrino sector to the lepton mixing up to permutations of the columns, since neutrino masses are unconstrained in the present framework.

The unitary matrix U_e from the charged lepton sector is determined by requiring invariance of $m_l^\dagger m_l$ under the subgroup G_e which is in general the product of cyclic groups Z_{m_i}, $i = 1, ..., p$. Denoting the generator of the cyclic group Z_{m_i} by Q_i, the invariance conditions read

$$Q_i^\dagger m_l^\dagger m_l Q_i = m_l^\dagger m_l \tag{44}$$

for all i. Since the set of generators Q_i distinguishes between the three generations of charged leptons, the unitary matrix U_e which simultaneously diagonalizes them is determined (up to permutations of columns and phases of the column vectors)

$$U_e^\dagger Q_i U_e = \hat{Q}_i \tag{45}$$

with $\hat{Q}_i$ diagonal. Plugging eq.(45) into eq.(44) shows that U_e also diagonalizes $m_l^\dagger m_l$, see eq.(5).

Finally we have

$$U_{PMNS} = U_e^\dagger(Q_i)\ \Omega(Z, X)\ R(\theta)\ K \quad , \tag{46}$$

up to permutations of rows and columns. We have spelled out the dependence on the choice of subgroups, which is specified by the generators of G_e and G_ν, and on the CP transformation, i.e. the set (Q_i, Z, X). In our approach neutrino and charged lepton masses remain as undetermined parameters and thus the ordering of rows and columns of the PMNS matrix is not fixed. For a given set (Q_i, Z, X) all three mixing angles, the Dirac phase and the two Majorana phases are determined in terms of the parameter θ that depends on the neutrino mass parameters m_{ij}. The Majorana phases are fixed up to the contribution from the matrix K which can only shift the phases by π.

Since the formalism is covariant under a change of basis, see eqs.(27,28), the mixing parameters do not depend on the particular basis used for the set of transformations (Q_i, Z, X). Especially, two sets of transformations which are related by a similarity transformation of the group G_f lead to the same physical results.

The reader may wonder why we have not assumed $X = \mathbb{1}$ from the very beginning, avoiding the change of basis described by Ω. The reason is that usually the representation (matrices) of the group G_f acting on lepton doublets is (are) explicitly given in model building. For a given representation $X = \mathbb{1}$ might not satisfy the constraint in eq. (26) or it might happen that this constraint allows for several independent solutions, leading to different results for the mixing parameters, as we will see in section 3.

Note that if X satisfies the relations in eqs. (2, 26,36), then also $e^{2i\gamma}X$ and ZX satisfy them. The first choice merely leads to an unphysical overall phase γ in the PMNS matrix, while the second one leads to the same PMNS matrix. This is because the same neutrino mass matrix that satisfies the conditions in eq. (39) with X also satisfies the conditions with ZX. Notice that X and $ZX = XZ^*$ form a pair of transformations and are transformed into each other by the mapping $Y \to ZY$.

Finally, we would like to comment on other possible symmetry breaking patterns involving CP as residual symmetry. We can consider CP to be unbroken in the charged

lepton sector, $G_e = Z_{m_1} \times \cdots \times Z_{m_p} \times CP$. Suitable relations among X and the generators of the cyclic groups have to be satisfied in this case, to consistently define G_e. As long as $Z_{m_1} \times \cdots \times Z_{m_p}$ distinguishes among the three generations, the presence of the additional CP symmetry does not have any further constraining effect. If instead two of the generations have the same transformation properties under $Z_{m_1} \times \cdots \times Z_{m_p}$, the symmetry G_e determines the mixing of the charged lepton sector up to a free parameter θ. In this case, one might wonder whether it is straightforward to accommodate the hierarchy among the charged lepton masses without a tuning of the parameters of the mass matrix m_l. Similar considerations can be made, if neutrinos are Dirac particles and $G_\nu = Z_{m_1} \times \cdots \times Z_{m_p} \times CP$.

2.4 Accidental CP symmetries

We conclude this section with some remarks about possible accidental CP symmetries and the conditions that the mass matrices must fulfill in order to yield non-trivial Dirac or Majorana phases (i.e. phases different from 0 or π). These conditions are useful to understand the results of the example $G_f = S_4$ that we present in the next section. We have imposed CP conservation in the neutrino, but not in the charged lepton sector. Thus, we might be led to the conclusion that non-trivial Dirac and Majorana phases are always generated in our approach. Actually this is not the case. Trivial phases are found when the mass matrices $m_l^\dagger m_l$ and m_ν, constrained by our choice of (Q_i, Z, X), satisfy the invariance conditions

$$Y^* m_l^\dagger m_l Y = (m_l^\dagger m_l)^* \quad , \qquad\qquad Y m_\nu Y = m_\nu^* \quad , \tag{47}$$

for some unitary symmetric matrix Y, to which we refer as an accidental CP symmetry. We have already seen that the general solution to the above invariance conditions is given by

$$Y_e = U_e^\dagger Y U_e^* \quad , \qquad Y_\nu = U_\nu^\dagger Y U_\nu^* \quad , \tag{48}$$

where Y_e and Y_ν are of the form shown in eqs. (8,9) in the mass basis. It is useful to express these conditions in terms of (Q_i, Z, X). As one can see, Y satisfies the first equality of eq. (48) if and only if

$$Q_i Y - Y Q_i^T = 0 \tag{49}$$

for all i. This condition ensures that Y is diagonal in the same basis as Q_i, that is Y_e is diagonal. Notice that if Y satisfies eq. (49) also $Y \prod_{i=1}^{p} (Q_i^*)^{n_i}$, $0 \leq n_i \leq m_i$, does so. Similarly, one can check that the second equality of eq. (48) implies

$$ZY - YZ^* = 0 \qquad\qquad XY^* - YX^* = 0 \quad . \tag{50}$$

The first equality is of the same form as the requirement for having a direct product of $Z_2 \times CP$, see eq.(36), while the second equality states that the two CP transformations X and Y commute. These conditions are, however, only necessary but not sufficient to ensure that Y_ν is diagonal and real. Indeed in the basis indicated by a hat in which $\hat{Z}$ is diagonal and $\hat{X} = \mathbb{1}$, the equalities in eq. (50) are satisfied by any real matrix $\hat{Y}$ commuting with $\hat{Z}$. Such a matrix is symmetric, orthogonal (Y being unitary) and block-diagonal. The application of the rotation $R(\theta)$ and of the transformation K (necessary in order to

transform to a basis in which m_ν is diagonal and positive (semi-)definite) still leaves Y_ν block-diagonal.

We can therefore distinguish the following cases: *a*) we cannot find a CP symmetry Y which fulfills eqs.(47-50). Then we have to expect non-trivial Dirac and Majorana phases; *b*) a CP symmetry Y exists which fulfills these equations and it is furthermore real and diagonal in the (neutrino) mass basis. Then all CP phases are trivial, see eqs. (17,20); *c*) we find a CP transformation Y which fulfills eq.(49) and is diagonal in the (neutrino) mass basis; however, it is not real in this basis, i.e. Y satisfies the first part of eq.(50), but not the second one. Then Y is not a CP symmetry of the setup, but it still leaves $m_l^\dagger m_l$ and $m_\nu^\dagger m_\nu$ invariant, as it is the case for Dirac neutrinos, compare eq.(21). Then we know $J_{CP}=0$ and $\sin\delta=0$. Furthermore, we can show that $U_{PMNS,ij}e^{-i(x_i-x_j^\nu)}=U^*_{PMNS,ij}$, see eq.(22), implies that

$$|\sin\alpha|=|\sin(x_1^\nu-x_2^\nu)| \quad , \quad |\sin\beta|=|\sin(x_1^\nu-x_3^\nu)| \tag{51}$$

with x_i^ν being the phases of the diagonal entries of Y_ν.[8]

In a setup characterized by (Q_i, Z, X) one natural candidate for accidental symmetries is $Y=ZX$ which always satisfies eq. (50) and the corresponding Y_ν is real and diagonal.

3 Example S_4 and CP

We analyze here the mixing patterns originating from the breaking of S_4 and CP to $G_\nu = Z_2\times CP$ and to G_e being an abelian subgroup of S_4. We choose S_4, since it is among the smallest discrete groups with an irreducible three-dimensional representation and it is well-known to be the smallest symmetry group which leads to TB mixing, if broken in a non-trivial way [4]. We first present the group S_4 in a basis convenient for our purposes and then discuss in detail the findings of our comprehensive study of the group S_4 and CP.

3.1 Group theory of S_4

The group S_4 can be defined in terms of three generators S, T and U [27] which fulfill the following relations

$$S^2=E\ ,\ \ T^3=E\ ,\ \ U^2=E\ , \tag{52}$$

$$(ST)^3=E\ ,\ \ (SU)^2=E\ ,\ \ (TU)^2=E\ ,\ \ (STU)^4=E \tag{53}$$

with E being the neutral element of S_4. Note that the generators S and T alone give rise to the group A_4. In the following we are only interested in the two irreducible (faithful) three-dimensional representations, called **3** and **3′**, to which we assign the three generations of left-handed leptons. We choose real representation matrices S, T and U (called in the same way as the abstract elements of the group S_4) for the representation **3′** (following the notation of [27]):

$$S=\begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} \quad , \quad T=\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ -1 & \sqrt{2} & -1\end{pmatrix} \quad , \quad U=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} \ . \tag{54}$$

[8]Notice that we have to assume that neither the solar nor the reactor mixing angle are 0 or $\pi/2$.

The representation matrices of the other three-dimensional representation **3** read S, T and $-U$.[9]

The group S_4 has Z_2, Z_3, Z_4 and $Z_2 \times Z_2$ as abelian subgroups. The nine Z_2 symmetries are generated by the elements

$$S, TST^2S, T^2STS, U, US, UT, UT^2, USTS, UST^2S\,, \tag{55}$$

while the generators of the four Z_3 symmetries can be chosen as

$$T\ ,\ \ ST\ ,\ \ ST^2\ ,\ \ TST\ , \tag{56}$$

and those of the three Z_4 symmetries as

$$STU\ ,\ \ UTS\ ,\ \ UT^2ST\ . \tag{57}$$

The Z_2 generating elements are divided into two classes (the first three of the list in eq.(55) and the last six ones), while the Z_3 subgroups as well as the Z_4 subgroups of S_4 are all conjugate to each other. There are four Klein subgroups. One of them, called K_N, is normal, while the three other ones K_i, $i = 1, 2, 3$ are conjugate to each other. Possible sets of generators of the different Klein subgroups are

$$K_N\ :\ S, TST^2S\ \ ,\ \ K_1\ :\ S, U\ \ ,\ \ K_2\ :\ TST^2S, UT^2\ \ ,\ \ K_3\ :\ T^2STS, UT\ \ . \tag{58}$$

3.2 Mixing patterns from flavour groups S_4 and A_4 without CP

We briefly repeat the mixing patterns which can be derived from the group S_4 as well as A_4, if the group G_ν is a Klein group and G_e an abelian subgroup of G_f, capable to distinguish the three generations of charged leptons. All these results can be found in [7, 28]. We have three different viable choices for the group G_e if $G_f = S_4$: $G_e = Z_3$, $G_e = Z_4$ and $G_e = Z_2 \times Z_2$. As is well-known, the choice $G_e = Z_3$ leads to TB mixing [2] (for example, we can choose T as generator for G_e and S and U for G_ν), while $G_e = Z_4$ (one possible choice of generator is STU) and $G_e = Z_2 \times Z_2$ (for example generated by the elements TST^2S and UT^2) give rise to bimaximal (BM) mixing [29]. In the case of $G_f = A_4$, we have a unique choice for $G_\nu = Z_2 \times Z_2$ (to be generated by S and TST^2S) and as choice for G_e only $G_e = Z_3$ (for example, generated by T). The mixing pattern is given by the familiar democratic mixing matrix, in which all elements have the same absolute value and the mixing parameters read $\sin^2\theta_{13} = 1/3$, $\sin^2\theta_{12} = 1/2$, $\sin^2\theta_{23} = 1/2$, $|J_{CP}| = 1/(6\sqrt{3})$ and $|\sin\delta| = 1$. This matrix has already been discussed many years ago as possible lepton mixing matrix [30].

[9] Notice that our choice of basis for S, T and U is related to the one of [27] with $\bar{S}$, $\bar{T}$ and $\bar{U}$, through the unitary transformation V

$$V = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -i/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & i/\sqrt{2} \end{pmatrix}$$

so that

$$S = V^\dagger \bar{S} V\ \ ,\ T = V^\dagger \bar{T} V\ \ ,\ U = V^\dagger \bar{U} V\ .$$

3.3 Results for $G_f = S_4$ and CP

We show the results of a comprehensive study in which we assume $G_f = S_4$, $G_\nu = Z_2 \times CP$ and G_e being $G_e = Z_3$, $G_e = Z_4$ or $G_e = Z_2 \times Z_2$. In order to facilitate understanding we present our results in terms of examples for the different cases. We find that it is sufficient to consider only a small number of cases which lead to different results for mixing angles and CP phases, since other possible choices of Q_i, Z and X are related by similarity transformations -belonging to the group S_4- to our representative solutions and thus cannot lead to new results. We concentrate in our discussion on the representation $\mathbf{3}'$ of S_4. However, if we assigned the three generations of left-handed leptons to the representation $\mathbf{3}$ instead, the results would be the same and no additional results would be found, because the generators of the triplet $\mathbf{3}$ just differ in the overall sign of the generator U from those of the representation $\mathbf{3}'$. It follows a short subsection with technical details necessary for the derivation of the results.

3.3.1 Choice of (Q_i, Z, X)

All possible choices of Z and X are related through similarity transformations (contained in S_4) to the following three Z

$$Z = S \ , \quad Z = SU \quad \text{and} \quad Z = U \tag{59}$$

and their corresponding X_i which fulfill the requirements stated in eqs.(2,26,36) of section 2. For all Z

$$X_1 = \mathbb{1} \ , \quad X_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \quad X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ ,$$
$$X_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{60}$$

are admissible and for $Z = S$ in addition

$$X_5 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \ , \quad X_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ . \tag{61}$$

As mentioned in section 2, these transformations are defined up to an overall phase. X_1 is the canonical CP transformation. Notice that in our particular basis the different X_i turn out to be proportional to elements of the group S_4: $X_2 \propto S$, $X_3 \propto U$, $X_4 \propto SU$, $X_5 \propto TST^2S$ and $X_6 \propto T^2STS$. However, this is just a coincidence and in general the transformations X_i do not belong to the flavour group G_f. We list the transformations Ω_i

which bring the different X_i into the canonical form $\tilde{X} = \mathbb{1}$, see eqs.(28,37):

$$\Omega_1 = \mathbb{1} \; , \;\; \Omega_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix} \; , \;\; \Omega_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix} \; , \;\; \Omega_4 = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; ,$$
$$\Omega_5 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & i \\ 0 & \sqrt{2}\,i & 0 \\ 1 & 0 & i \end{pmatrix} \; , \;\; \Omega_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 & 1 \\ 0 & \sqrt{2}\,i & 0 \\ i & 0 & 1 \end{pmatrix} \; . \tag{62}$$

We note that, obviously, for different choices of Z, but using the same form of the matrices Ω_i, the rotation matrix $R(\theta)$ defined in eq. (43) changes its form: it is a rotation in the (13)-plane for $Z = S$, a rotation in the (23)-plane for $Z = SU$ and for $Z = U$ a rotation in the (12)-plane.

The different choices of G_e, for which we discuss lepton mixing, can be represented by

$$\begin{array}{lll} Q = T & \text{for} & G_e = Z_3 \; , \\ Q = STU & \text{for} & G_e = Z_4 \; , \\ Q_1 = TST^2S \; , \;\; Q_2 = UT^2 & \text{for} & G_e = Z_2 \times Z_2 \; . \end{array} \tag{63}$$

The matrix U_e which diagonalizes the charged lepton mass matrix $m_l^\dagger m_l$ is then of the form

$$U_e = \begin{pmatrix} \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \quad \text{for } Q = T \; , \tag{64}$$

$$U_e = \begin{pmatrix} -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix} \quad \text{for } Q = STU \tag{65}$$

and

$$U_e = \begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix} \quad \text{for } Q_1 = TST^2S \text{ and } Q_2 = UT^2 \; . \tag{66}$$

3.3.2 Lepton mixing parameters

We have performed a comprehensive study in which we consider all possible choices of Q_i, Z and X and all possible permutations of columns and rows of the mixing matrix. However, we show in the following only results which we consider phenomenologically interesting in the sense that we can find a value of the parameter θ such that the resulting mixing angles are reasonably close to their best fit values which we take from [6]. As measure we use a χ^2 function defined in the usual way and require its minimal value to be less than 100. In this way, all solutions leading to vanishing θ_{13} independent of the parameter θ are excluded, since $\theta_{13} = 0$ is disfavored at the $10\,\sigma$ level by global fits, $\sin^2\theta_{13} = 0.023 \pm 0.0023$, [6]. We define two different χ^2 functions, because $\sin^2\theta_{23}$ has two best fit values $\sin^2\theta_{23} = 0.41$

and $\sin^2\theta_{23} = 0.59$.[10] For this reason, we display in the tables below in several occasions different best fit values θ_{bf} for the parameter θ for which the χ^2 function has a global minimum. In the tables we also display the results for the sines of the CP phases δ, α and β (and the Jarlskog invariant J_{CP}). These quantities are presented in terms of absolute values, since the sign of the Jarlskog invariant J_{CP} depends on the ordering of rows and columns, while the sign of $\sin\alpha$ and $\sin\beta$ depends on the CP parity of the neutrino states which is encoded in the matrix K, see eq.(42) in section 2 (changing CP parity shifts the Majorana phase by π).

Requiring $\chi^2 < 100$, we find five viable solutions (Case I, II, IV, V and the case with $G_e = Z_4$ or $G_e = Z_2 \times Z_2$). However, in all these cases the Majorana phases are trivial, i.e. $\sin\alpha = 0$ and $\sin\beta = 0$. Thus, we have included another case, called Case III, which leads to non-trivial Majorana phases depending on the parameter θ, although the minimum value of its χ^2 functions is above 100.

In the following we put special emphasis on the results for the CP phases δ, α and β, because frequently an accidental CP symmetry is present which leads to a trivial Dirac phase and/or Majorana phases.

We first take $G_e = Z_3$ and choose $Q = T$. Five different cases can be distinguished and are represented by

$$\begin{array}{ccc} \mathrm{I} & Z = S\ , & X = X_1 \\ \mathrm{II} & Z = S\ , & X = X_3 \\ \mathrm{III} & Z = S\ , & X = X_5 \\ \mathrm{IV} & Z = SU\ , & X = X_1 \\ \mathrm{V} & Z = SU\ , & X = X_2\ . \end{array} \tag{67}$$

Notice that in Case I X_2 is also a CP symmetry of the neutrino sector, since $ZX_1 = X_2$. Similarly, it holds in Case II that $X_4 = ZX_3$ is also a CP symmetry and in Case III it is $X_6 = ZX_5$. Analogously, we find that in Case IV $X_4 = ZX_1$ is also present and in Case V $X_3 = ZX_2$. Thus, the six possible choices of X_i for $Z = S$, mentioned in subsection 3.3.1, give rise to three independent cases and for $Z = SU$ the four possible X_i lead to effectively two different cases. In Cases I, II, IV and V we find a value of θ for which the computed mixing angles agree rather well with the ones obtained in global fits [6]. This is shown in table 1 together with the formulae for the mixing parameters in terms of generic θ.

In Case II and Case V all CP phases are trivial. Thus, an accidental CP symmetry Y common to the charged lepton and neutrino sector has to be present. Indeed, $Y = X_3$ is such an accidental symmetry in both cases, because it satisfies eq.(49) for $Q = T$ and, in the neutrino sector, it is once imposed as $X = X_3$ (Case II) and once not directly imposed, but $ZX_2 = X_3$ holds (Case V) [implying in both cases that the conditions in eq.(50) are fulfilled and Y is diagonal and real in the neutrino mass basis].

In Case I and Case IV $\sin\delta$ vanishes for $\sin 2\theta = 0$ and the Majorana phases are always trivial. This result can be traced back to the fact that X_3 is a CP symmetry in the charged lepton sector, it fulfills the conditions in eq.(50) and its form in the mass basis of the neutrinos is

$$Y_\nu = U_\nu^\dagger X_3 U_\nu^\star \quad \text{with} \quad U_\nu = \Omega_1 R(\theta) K\ , \tag{68}$$

[10]Note that the 1σ errors are not completely gaussian for $\sin^2\theta_{23}$. However, we use for the smaller best fit value a 1σ error of ± 0.031 and for the larger one an error of ± 0.022.

	I	II	IV	V
$\sin^2\theta_{13}$ $\sin^2\theta_{12}$ $\sin^2\theta_{23}$ $\lvert J_{CP}\rvert$ $\lvert\sin\delta\rvert$ $\sin\alpha$ $\sin\beta$	$\frac{2}{3}\sin^2\theta$ $\frac{1}{2+\cos 2\theta}$ $\frac{1}{2}$ $\frac{\lvert\sin 2\theta\rvert}{6\sqrt{3}}$ 1^a 0 0	$\frac{2}{3}\sin^2\theta$ $\frac{1}{2+\cos 2\theta}$ $\frac{1}{2}\left(1-\frac{\sqrt{3}\sin 2\theta}{2+\cos 2\theta}\right)$ 0 0 0 0	$\frac{1}{3}\sin^2\theta$ $\frac{\cos^2\theta}{2+\cos^2\theta}$ $\frac{1}{2}$ $\frac{\lvert\sin 2\theta\rvert}{6\sqrt{6}}$ 1^a 0 0	$\frac{1}{3}\sin^2\theta$ $\frac{\cos^2\theta}{2+\cos^2\theta}$ $\frac{1}{2}\left(1-\frac{2\sqrt{6}\sin 2\theta}{5+\cos 2\theta}\right)$ 0 0 0 0
θ_{bf}	0.185	0.184 $\theta_{23}<\pi/4$ 2.958 $\theta_{23}>\pi/4$	0.268	0.251 $\theta_{23}<\pi/4$ 2.907 $\theta_{23}>\pi/4$
χ^2_{min}	18.4 $\theta_{23}<\pi/4$ 26.7 $\theta_{23}>\pi/4$	10.3 $\theta_{23}<\pi/4$ 10.5 $\theta_{23}>\pi/4$	10.2 $\theta_{23}<\pi/4$ 18.5 $\theta_{23}>\pi/4$	16.1 $\theta_{23}<\pi/4$ 27.2 $\theta_{23}>\pi/4$
$\sin^2\theta_{13}(\theta_{\mathrm{bf}})$	0.023	0.022	0.023	0.021 $\theta_{23}<\pi/4$ 0.018 $\theta_{23}>\pi/4$
$\sin^2\theta_{12}(\theta_{\mathrm{bf}})$	0.341	0.341	0.317	0.319 $\theta_{23}<\pi/4$ 0.321 $\theta_{23}>\pi/4$
$\sin^2\theta_{23}(\theta_{\mathrm{bf}})$	0.5	0.394 $\theta_{23}<\pi/4$ 0.606 $\theta_{23}>\pi/4$	0.5	0.299 $\theta_{23}<\pi/4$ 0.688 $\theta_{23}>\pi/4$
$\lvert J_{CP}\rvert(\theta_{\mathrm{bf}})$	0.0348	0	0.0348	0

Table 1: Results for the mixing parameters in terms of the parameter θ for the Cases I, II, IV and V. We display the best fit value θ_{bf} for θ for which the χ^2 functions have a global minimum χ^2_{min}. Since the global fit [6] gives two best fit values for $\sin^2\theta_{23}$, one with $\theta_{23}<\pi/4$ and one with $\theta_{23}>\pi/4$, we distinguish these two possibilities. Since the CP phases are independent of θ, we only give the values of mixing angles and the absolute value of J_{CP} for $\theta=\theta_{\mathrm{bf}}$. Note that the fundamental interval of θ is the range between 0 and π.
[a] In the special case $\sin 2\theta=0$ $\sin\delta$ vanishes, since then J_{CP} vanishes. However, then the mixing angles are in considerable disagreement with the best fit values from [6].

which explicitly reads for Case I and IV

$$Y_{\nu,\mathrm{I}} = K^\star \begin{pmatrix} \cos 2\theta & 0 & \sin 2\theta \\ 0 & 1 & 0 \\ \sin 2\theta & 0 & -\cos 2\theta \end{pmatrix} K^\star \quad \text{and} \quad Y_{\nu,\mathrm{IV}} = K^\star \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta & -\cos 2\theta \end{pmatrix} K^\star \,, \tag{69}$$

respectively. For the special choice $\sin 2\theta = 0$ (and thus $\cos 2\theta = \pm 1$) these matrices are diagonal and real and thus X_3 is an accidental CP symmetry of the charged lepton and neutrino mass matrices and consequently all CP phases are trivial.

As regards the mixing angles, the atmospheric mixing turns out to be maximal in Case I and IV, because the elements of the second and third rows of the PMNS matrix have the same absolute values, i.e. $|U_{\mu i}| = |U_{\tau i}|$ for $i = 1,2,3$ [9, 10, 31]. This feature also

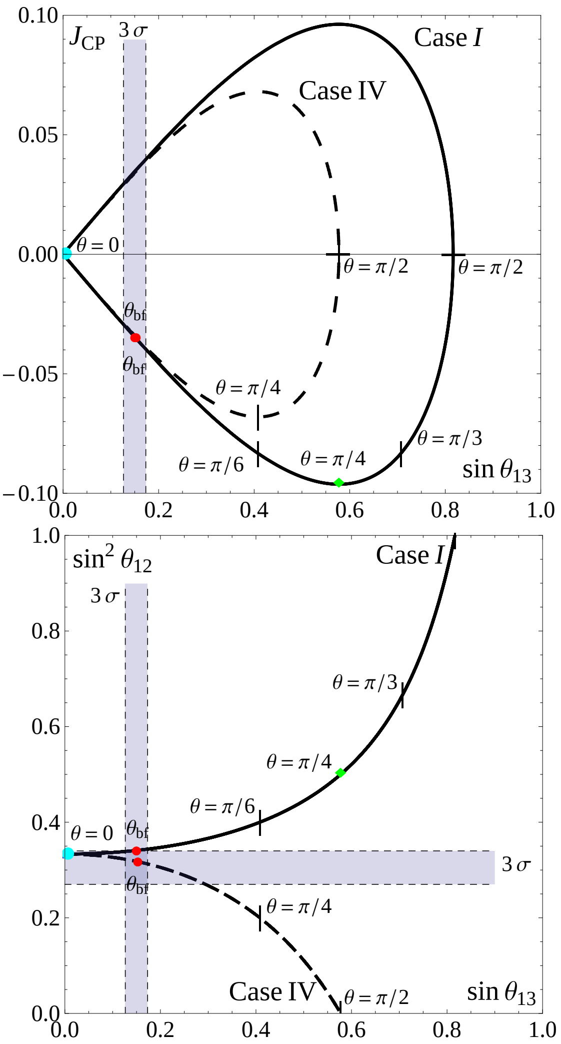


Figure 1: Results for the mixing parameters $\sin\theta_{13}$, $\sin^2\theta_{12}$ and J_{CP} for Case I (straight line) and Case IV (dashed line). We mark the value $\theta_{\rm bf}$ of the parameter θ for which the χ^2 functions have a global minimum with a red dot. We also indicate special points $\theta = 0$ and $\theta = \pi/4$ for which the second Z_2 symmetry in the neutrino sector becomes part of the group S_4 and well-known mixing patterns like TB mixing and the democratic mixing are reproduced. For better guidance of the eye we also mark $\theta = \pi/n$, $n = 6, 4, 3, 2$ on the curves. The shown $3\,\sigma$ ranges for the mixing angles are taken from [6].

explains the maximal Dirac phase (as long as $\sin\theta_{13} \neq 0$ and $\sin 2\theta_{12} \neq 0$). In Case I and Case II $Z = S$ enforces the second column of the PMNS matrix to be tri-maximal. As a consequence, the solar mixing angle has a lower limit given by $\sin^2\theta_{12} \geq 1/3$ [32], which is disfavored by the global fits [6], $\sin^2\theta_{12} = 0.30 \pm 0.013$, at the $2\,\sigma$ level. On the other hand, in Case IV and Case V $Z = SU$ leads to a lepton mixing matrix whose first column coincides with the one of TB mixing and consequently the solar mixing angle has an upper limit given by $\sin^2\theta_{12} \leq 1/3$ [33]. The results of Case I and Case II have been discussed previously in the literature as generalization of TB mixing [9]. In Case I and Case IV maximal atmospheric mixing and a maximal Dirac phase can be achieved by imposing as one of the symmetries of the neutrino mass matrix the so-called $\mu\tau$ reflection symmetry (in the charged lepton mass basis) which is a generalized CP transformation [9, 10]. The mixing parameters in the different cases are illustrated in figures 1 and 2. For a better presentation we display $\sin\theta_{13}$ instead of its square.[11] We show the reactor and the solar

[11]We take as best fit value of $\sin\theta_{13}$ $\sin\theta_{13} = 0.15$ and as $3\,\sigma$ range $0.126 \leq \sin\theta_{13} \leq 0.173$. These

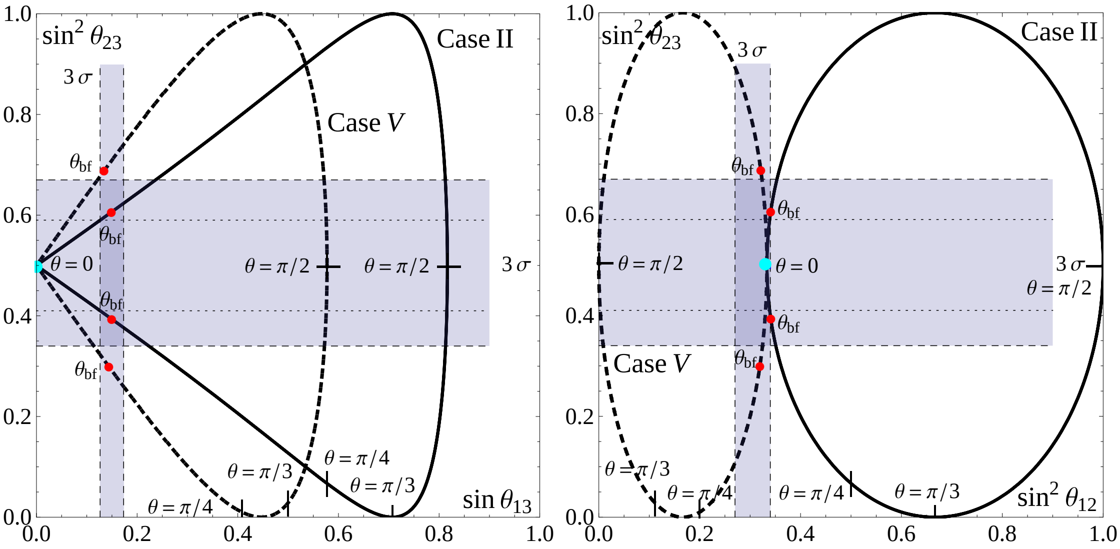


Figure 2: Results for the atmospheric, solar and reactor mixing angles for Case II (straight line) and Case V (dashed line). We mark the value $\theta_{\rm bf}$ of the parameter θ for which the χ^2 functions have a global minimum with a red dot. We also indicate the special point $\theta = 0$ for which the second Z_2 symmetry in the neutrino sector becomes part of the group S_4 and TB mixing is reproduced. For better guidance of the eye we also mark $\theta = \pi/n$, $n = 4, 3, 2$ on the curves. The shown $3\,\sigma$ ranges for the mixing angles and the best fit values of the atmospheric mixing angle are taken from [6]. Notice that the plot for $\sin^2\theta_{12}$ and $\sin\theta_{13}$ is the same as for Case I and Case IV and can be found in figure 1.

mixing angles and J_{CP} for Case I and Case IV, while we present the three mixing angles for Case II and Case V. Notice that the plot in the plane of $\sin^2\theta_{12}$ and $\sin\theta_{13}$ is the same in Case I (IV) and Case II (V) . Furthermore, we mark the best fit value $\theta_{\rm bf}$ for which the χ^2 functions have a global minimum with a red dot. For better guidance of the eye we also indicate the values $\theta = \pi/n$, $n = 6, 4, 3, 2$ on the curves.

As we have mentioned, we discuss Case III because it is the only one in which all CP phases are in general non-trivial and depend on the parameter θ, although in this case the minimum value of the χ^2 functions is (slightly) larger than 100. The results are collected in table 2. Again, $Z = S$ is responsible for the fact that the second column of the PMNS matrix is tri-maximal and thus the solar mixing angle has a lower limit $\sin^2\theta_{12} \geq 1/3$. Furthermore, we note that the solar and the atmospheric mixing angles are closely related because their sine squares are either equal or fulfill $\sin^2\theta_{12} = 1 - \sin^2\theta_{23}$.

The best fit value $\theta_{\rm bf}$ is $\pi/4$ because for this value the result for $\sin^2\theta_{13}$ is minimized: $\sin^2\theta_{13}(\theta_{\rm bf}) = (2-\sqrt{3})/6 \approx 0.045$. Notice that for $\theta_{\rm bf} = \pi/4$ the Dirac phase δ is trivial and also one of the Majorana phases. The former is trivial, because of a common CP symmetry $Y = X_3$ of the matrices $m_l^\dagger m_l$ and $m_\nu^\dagger m_\nu$, as explained in subsection 2.4. This CP transformation fulfills eq.(49) for $Q = T$ and its form in the mass basis of the neutrinos

values are derived from [6].

	III	
$\sin^2\theta_{13}$	$\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\sin 2\theta\right)$	
$\sin^2\theta_{12}$	$\frac{2}{4+\sqrt{3}\sin 2\theta}$	
$\sin^2\theta_{23}$	$\frac{2}{4+\sqrt{3}\sin 2\theta}$	$1-\frac{2}{4+\sqrt{3}\sin 2\theta}$
$\lvert J_{CP}\rvert$	$\frac{\lvert\cos 2\theta\rvert}{6\sqrt{3}}$	
$\lvert\sin\delta\rvert$	$\left\lvert\frac{(4+\sqrt{3}\sin 2\theta)\cos 2\theta\sqrt{4-2\sqrt{3}\sin 2\theta}}{5+3\cos 4\theta}\right\rvert$	
$\lvert\sin\alpha\rvert$	$\left\lvert\frac{\sqrt{3}+2\sin 2\theta}{2+\sqrt{3}\sin 2\theta}\right\rvert$	
$\lvert\sin\beta\rvert$	$\left\lvert\frac{4\sqrt{3}\cos 2\theta}{5+3\cos 4\theta}\right\rvert$	
θ_{bf}	0.785 $\theta_{23}<\pi/4$	0.785 $\theta_{23}>\pi/4$
χ^2_{min}	106.7	110.5
$\sin^2\theta_{13}(\theta_{\text{bf}})$	0.045	
$\sin^2\theta_{12}(\theta_{\text{bf}})$	0.349	
$\sin^2\theta_{23}(\theta_{\text{bf}})$	0.349	0.651
$\lvert J_{CP}\rvert(\theta_{\text{bf}})$	0	
$\lvert\sin\delta\rvert(\theta_{\text{bf}})$	0	
$\lvert\sin\alpha\rvert(\theta_{\text{bf}})$	1	
$\lvert\sin\beta\rvert(\theta_{\text{bf}})$	0	

Table 2: Results for the mixing parameters in Case III. As one can see this is the only case with a non-trivial dependence of the CP phases on the parameter θ. The possibility to exchange the second and third rows of the PMNS matrix gives rise to the two solutions which differ in their result for $\sin^2\theta_{23}$. The χ^2 functions have a global minimum with $\chi^2 \gtrsim 100$ at $\theta_{\text{bf}} \approx \pi/4$. Also in this case it is sufficient to consider values of θ between 0 and π.

is

$$Y_{\nu,\text{III}} = U_\nu^\dagger X_3 U_\nu^\star = K^\star \begin{pmatrix} -i\sin 2\theta & 0 & i\cos 2\theta \\ 0 & -1 & 0 \\ i\cos 2\theta & 0 & i\sin 2\theta \end{pmatrix} K^\star \,. \tag{70}$$

This matrix is for generic values of the parameter θ neither diagonal nor real. However, if $\cos 2\theta = 0$ (as is true for $\theta = \pi/4$), $Y_{\nu,\text{III}}$ becomes diagonal. Since its entries are not real (independent of K), X_3 is not a symmetry of the neutrino mass matrix itself but only of $m_\nu^\dagger m_\nu$ and consequently Majorana phases are not trivial in general. According to eq. (51), they can be directly read off from $Y_{\nu,\text{III}}$: $|\sin\alpha| = 1$ and $\sin\beta = 0$.

For the choice $G_e = Z_4$ we find only one case which passes our selection criteria. We choose -as mentioned above- as representative Z_4 generating element $Q = STU$ and take for Z and X

$$Z = U \quad \text{and} \quad X = X_2 \,. \tag{71}$$

Notice that the same results would be obtained with the CP symmetry X_4, since $X_4 =$

ZX_2. The results for the mixing parameters and the best fit values $\theta_{\rm bf}$ of the parameter θ can be found in table 3 and the mixing angles are displayed in figure 3 as well. We

	$G_e = Z_4$ or $G_e = Z_2 \times Z_2$	
	Case a	Case b
$\sin^2\theta_{13}$	$\frac{1}{4}\left(\sqrt{2}\cos\theta + \sin\theta\right)^2$	
$\sin^2\theta_{12}$	$\frac{2}{5-\cos 2\theta - 2\sqrt{2}\sin 2\theta}$	
$\sin^2\theta_{23}$	$\frac{4\sin^2\theta}{5-\cos 2\theta - 2\sqrt{2}\sin 2\theta}$	$1 - \frac{4\sin^2\theta}{5-\cos 2\theta - 2\sqrt{2}\sin 2\theta}$
J_{CP}	0	
$\sin\alpha$	0	
$\sin\beta$	0	
$\theta_{\rm bf}$	2.009 $\theta_{23} < \pi/4$	2.010 $\theta_{23} > \pi/4$
$\chi^2_{\rm min}$	11.6	11.7
$\sin^2\theta_{13}(\theta_{\rm bf})$	0.023	
$\sin^2\theta_{12}(\theta_{\rm bf})$	0.256	
$\sin^2\theta_{23}(\theta_{\rm bf})$	0.420	0.581

Table 3: Results for the mixing parameters in terms of θ for the only viable case which we can find for $G_e = Z_4$ or $G_e = Z_2 \times Z_2$. The possible exchange of the second and third rows of the PMNS matrix gives rise to the two different solutions, Case a and Case b. All CP phases are trivial independently of the parameter θ. We also display the mixing parameters at $\theta_{\rm bf}$, best fit points for which the χ^2 functions have a global minimum. Again, the parameter θ can be chosen in the interval between 0 and π.

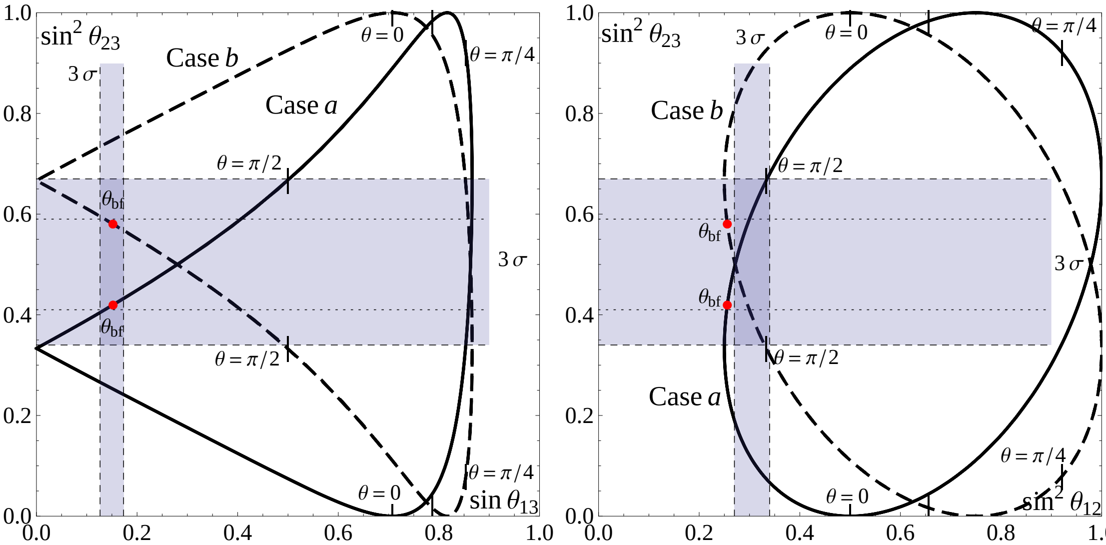
notice the existence of two solutions, Case a and Case b, which arise from the freedom to exchange the second and the third rows of the PMNS matrix. The χ^2 function of Case a has a global minimum with $\theta_{23}(\theta_{\rm bf}) < \pi/4$, while that of Case b has a global minimum with $\theta_{23}(\theta_{\rm bf}) > \pi/4$. These best fit points $\theta_{\rm bf}$ are indicated as red dots in figure 3. In addition, we mark certain values of θ in order to guide the eye.

CP phases are trivial independent of the parameter θ which indicates the presence of an accidental CP symmetry Y in the charged lepton and neutrino sector. Indeed, the CP symmetry $X = X_2$ of the neutrino sector is also a CP symmetry of the charged lepton mass matrix $m_l^\dagger m_l$, because $Q = STU$ and X_2 fulfill eq. (49).

As one can check, the PMNS matrix has one column of the form $(1/2, 1/\sqrt{2}, 1/2)^T$ or $(1/2, 1/2, 1/\sqrt{2})^T$ which is in common (up to permutation) with the BM mixing pattern originating from the charged lepton sector (see eq.(65) for $G_e = Z_4$ and eq.(66) for $G_e = Z_2 \times Z_2$). Patterns with such a column have been recently mentioned in [34].

If a Klein group is preserved in the charged lepton sector and $G_\nu = Z_2 \times CP$, we find the same results as for G_e being a Z_4 symmetry. One representative choice of the generators of the Klein group in the charged lepton sector is

$$Q_1 = TST^2S \quad \text{and} \quad Q_2 = UT^2 \tag{72}$$



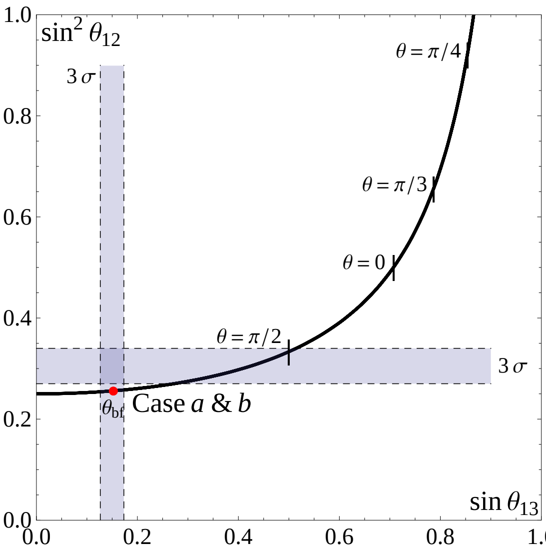


Figure 3: Results for the mixing angles $\sin\theta_{13}$, $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$ for Case a (straight line) and Case b (dashed line) for $G_e = Z_4$ or $G_e = Z_2 \times Z_2$. We mark the value θ_{bf} of the parameter θ for which the χ^2 functions have a global minimum with a red dot. For better guidance of the eye we also mark $\theta = 0, \pi/4, \pi/3, \pi/2$ on the curves. The shown $3\,\sigma$ ranges for the mixing angles and the best fit values of the atmospheric mixing angle are taken from [6].

and for Z and X of the neutrino sector we take

$$Z = U \quad \text{and} \quad X = X_1 \,. \tag{73}$$

All CP phases are trivial, because the imposed CP symmetry of the neutrino sector $X = X_1$, the canonical CP symmetry, is also present in the charged lepton sector. As one can check, the two generators Q_1 and Q_2 fulfill: $Q_i = Q_i^T$ with $i = 1, 2$ as required for $X = X_1$ by the constraint in eq.(49). The choice of X_3 as CP symmetry in the neutrino sector leads to the same results as the choice of X_1, since $Z X_1 = X_3$.

3.4 Symmetry enhancement for particular values of θ

As has been mentioned at the beginning of subsection 2.3, the maximal symmetry of a neutrino mass matrix m_ν for three generations of Majorana neutrinos is a Klein group, if neutrino masses are unconstrained. In the present approach we assume in general only one

of these Z_2 symmetries to be a subgroup of the flavour symmetry $G_f = S_4$. (This is similar to the approach in [8].) However, we notice that for the various cases presented particular values of the parameter θ (in general different for the different cases) exist for which the second Z_2 symmetry of the neutrino mass matrix m_ν can be promoted to a subgroup of a finite flavour group which contains G_f. Thus, in these cases the result of the mixing angles (and the CP phases) can be achieved through the symmetry breaking of this finite flavour group to a Klein group (and the symmetry CP) in the neutrino sector instead to $Z_2 \times CP$.

In all cases displayed in table 1 the limit $\theta \to 0$ corresponds to TB mixing (and trivial CP phases). In this limit the second Z_2 symmetry of the neutrino sector turns out to be generated by U, belonging to S_4, so that the residual symmetry in the neutrino sector is $Z_2 \times Z_2 \times CP$ instead of only $Z_2 \times CP$. The results of the mixing angles are thus the same as in the case without a CP symmetry in the neutrino sector, see subsection 3.2. In Case I also the limit $\theta \to \pi/4$ is noteworthy in which the mixing parameters take the values known from the democratic mixing matrix. This can be understood, because the second Z_2 symmetry of the neutrino mass matrix is now generated by TST^2S so that again the neutrino sector is invariant under a residual symmetry $Z_2 \times Z_2 \times CP$. As mentioned in subsection 3.1, S and T generate the group A_4 and according to subsection 3.2 the resulting mixing pattern is the democratic one. These special points are marked in figures 1 and 2 with a cyan dot and a green diamond, respectively.

Interestingly, in Case II the PMNS matrix is of the form

$$U_{PMNS} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix} R(\theta)\, K \, . \tag{74}$$

This result should be compared with the findings of [7] in which it has been shown that the PMNS matrix takes this form (with a matrix K with arbitrary phases, because Majorana phases are not determined in that approach) for $\theta = \pm\pi/12$ and $\theta = \pm\pi/24$, if the groups $\Delta(96)$ and $\Delta(384)$ are broken to a Z_3 symmetry in the charged lepton sector and to a Klein group in the neutrino sector, respectively.

For Case III note that the results of mixing angles and the Dirac phase coincide for the best fit point $\theta_{\rm bf} = \pi/4$ with those found in the case of a flavour group $\Delta(96)$ being broken to a Klein group in the neutrino sector and to a Z_3 symmetry in the charged lepton one [7]. Indeed, we can check that for $\theta = \pi/4$ the second Z_2 symmetry under which the neutrino mass matrix m_ν is invariant can be generated by

$$\check{Z} = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix} . \tag{75}$$

Obviously, this matrix does not represent an element of the group S_4 in our real basis for S, T and U. If we consider the group generated by S, T, U and $\check{Z}$, we find a group of order 96, which has as subgroup S_4. It might be that this group is isomorphic to $\Delta(96)$, but we have not performed a detailed analysis. Another particular value of the parameter θ is $\theta = 0$, because in this case the element TST^2S is promoted to a Z_2 generator under whose action the neutrino mass matrix is invariant (independent of any CP transformation). Both the

residual symmetry $G_e = Z_3$ in the charged lepton and $Z_2 \times Z_2$ in the neutrino sector are then generated through elements written in terms of S and T only so that the relevant flavour group is A_4 rather than S_4. As has been recapitulated in subsection 3.2, the flavour group A_4 broken in this way leads to the democratic mixing pattern which coincides for $\theta = 0$ with the results of Case III.

For $G_e = Z_4$ one particular limit is given by $\theta \to 0$ for which the (absolute values of the) PMNS matrix of Case a and Case b take(s) the form

$$||U_{PMNS,a}|| = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & 1/2 & 1/\sqrt{2} \end{pmatrix} \;, \;\; ||U_{PMNS,b}|| = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \;, \tag{76}$$

respectively. As one recognizes, this is the BM mixing matrix up to permutations of rows and columns. From the viewpoint of group theory the value $\theta = 0$ allows to promote S to the generator of one of the Z_2 symmetries of the matrix m_ν and thus the Klein group containing S and U is conserved in the neutrino sector. As repeated in subsection 3.2, the choice $G_f = S_4$, $G_e = Z_4$ and G_ν being a Klein group leads to BM mixing. If G_e is also a Klein group, very similar statements hold: namely, $\theta = 0$ promotes S to the generator of a Z_2 symmetry of the matrix m_ν and the breaking of $G_f = S_4$, $G_{e,\nu}$ being (non-normal) Klein groups with $G_e \neq G_\nu$ leads to BM mixing.

3.5 Comments on group structure of S_4 and CP

As has been mentioned in subsection 2.2, the group structure of the combination of a flavour symmetry G_f and a generalized CP symmetry is in general the semi-direct product of the group G_f and the Z_2 symmetry associated with the CP transformation X. For particular cases however the product can be direct. For $G_f = S_4$ and the various X_i, see eqs.(60, 61), only the canonical CP transformation $X = X_1$ leads to a direct product $S_4 \times CP$, because only in this case $AX = XA$ holds for all generators A (and thus all elements) of S_4 (we have used the fact that all representation matrices are real). In all the other cases $AX = XA$ is only satisfied for one or two of the three generators S, T and U. In particular, for $X = X_2 \propto S$, $X = X_3 \propto U$ and $X = X_4 \propto SU$ the relation $AX = XA$ obviously holds for $A = S$ and $A = U$, whereas for $A = T$ we find $X^{-1}TX = STS$, T^2 and TST, respectively. For the two CP transformations $X = X_5$ and $X = X_6$ $AX = XA$ is only fulfilled for the generator $A = S$, as required in order to preserve in the neutrino sector the direct product of the Z_2 symmetry generated by S and of the CP transformation X, see eq.(36). For $A = U$ we find instead $X^{-1}AX = SU$ in both cases and $A = T$ gives rise to $X^{-1}AX = TS$ and ST, respectively.

3.6 Comments on $G_f = A_4$

We can easily deduce results for $G_f = A_4$ from those obtained for $G_f = S_4$. As mentioned in subsection 3.1 the generators S and T alone give rise to A_4. Thus, the only thing to do is to consider the above results which can be obtained with generators Q_i and Z made up of S and T. This constrains us to take G_e as Z_3 symmetry. Furthermore, only Case I, II

and III are admissible, if Z is required to be a product of S and T. From the viewpoint of A_4 Case I and Case III are qualitatively different from Case II, since in the latter case X is not proportional to an element of A_4, while it is for Case I and Case III.

4 Summary and conclusions

Motivated by the recent measurement of the reactor mixing angle θ_{13}, we have considered a framework with three Majorana neutrinos in which a flavour symmetry G_f and a generalized CP transformation are combined and are broken in the neutrino sector to $G_\nu = Z_2 \times CP$, $Z_2 \subset G_f$, and in the charged lepton one to $G_e \subset G_f$. We have shown that several conditions have to be fulfilled in order to consistently define such a framework. These conditions constrain the possible choice of CP transformations. The mathematical structure of the group arising from G_f and CP is a semi-direct product of the form $G_{CP} = G_f \rtimes H_{CP}$. We have analyzed the lepton mixing for a general G_{CP} and have found that all mixing angles and all CP phases are determined in terms of a single real parameter θ, which can take values between 0 and 2π. In contrast to the approach with a flavour group G_f only we are also able to constrain Majorana phases in our framework. In order to show concrete examples and find new interesting mixing patterns, we have performed a comprehensive study for the group $G_f = S_4$. As expected, we have found several inequivalent CP transformations to be compatible with all requirements, which lead to different results for mixing angles and CP phases. Out of all possibilities only five give rise to phenomenologically interesting patterns, i.e. patterns for which the mixing angles are reasonably close to the best fit values for a certain value of the parameter θ. In particular, a non-vanishing value of the mixing angle θ_{13} can be easily accommodated and all mixing parameters are strongly correlated. Interestingly, four of these cases (Case I, II, IV and V) require G_e to be a Z_3 symmetry, while it is a Z_4 symmetry or a Klein group in the fifth case (see table 3). Among the interesting cases we also find two cases, Case I and Case IV, in which the neutrino mass matrix m_ν is invariant under $\mu\tau$ reflection symmetry (in the charged lepton mass basis). This particular generalized CP transformation has already been studied in the literature. Surprisingly, several of these cases lead to trivial CP phases due to the presence of accidental CP symmetries. For this reason, we have also discussed another case (Case III) in which all CP phases are (non-trivial) functions of the parameter θ. However, this case cannot accommodate the best fit values of the mixing angles so well, see table 2. In addition, we have studied particular values of θ for which the symmetry of the neutrino sector is enhanced to $G_\nu = Z_2 \times Z_2 \times CP$, with the Klein group belonging to a finite flavour group. Such cases are interesting because they allow us to re-cover results which have been derived with the help of a flavour symmetry G_f only. Since A_4 is a subgroup of S_4, we have also commented on cases for which $G_f = A_4$ is sufficient and have shown that three of the six cases (Case I, II and III) which we have studied for $G_f = S_4$ can also be reproduced with $G_f = A_4$.

It would be interesting to implement one of the presented cases in an actual model, for example in a supersymmetric context in which the breaking of the group G_{CP} to G_e and G_ν is spontaneous due to non-vanishing vacuum expectation values of some flavons. In such a model for Case I or Case IV the prediction of a maximal Dirac phase δ for

generic θ would not depend on the parameters of the theory and thus would represent a concrete realization of geometrical CP violation [14]. Apart from the usual challenge that corrections in a certain model can spoil some of the leading order predictions, models with such a framework have to face another challenge, namely the prediction of the parameter θ which depends on the entries of the neutrino mass matrix, see eq.(43), and whose size is important for accommodating the best fit values of the mixing angles well, compare tables 1-3.

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